THE MATHEMATICS TEACHER

VOLUME XIV DECEMBER 1921 NUMBER 8

RELIGIO MATHEMATICI*

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APOLOGIA

Well aware that I, to-day, break all precedents in the nature of a presidential address, it is proper that I offer due apology and confess my unconventionality. It is proper, too, that I should frankly say that I am aware that my message is wanting in the rigor of demonstration to which our science accustoms us; that it involves no mathematics beyond the merest elements; that it will be listened to with regret by some and with disapproval by others; and that it is likely to carry conviction to only a limited number.

Why, then, should I choose such a topic? Why should I force my friends to apologize for my thought; and why should I place others in a favorable position to condemn my action? The answer is a simple one:—I believe it to be a duty laid upon us, that one of our number should, on some occasion as this, say what I shall say. I regret that another,—one with more faith, one better endowed with power of expression, one whose words would carry greater weight than any I can utter,—might not have had the opportunity that is mine to deliver the message. And finally, by way of apology, I say with perfect candor that I speak not at all to the audience here present,—for what I shall say will convey nothing that is new to any one of you; I speak rather to those who are not present, those who are not of our special guild, and those to whom the teaching of mathematics may possibly, by this means, appear in a somewhat different, and I hope in a somewhat more favorable light.

*Presidential address delivered before the Mathematical Association of America, September 7, 1921. Printed in the "American Mathematical Monthly," October, 1921, in the "Teachers College Record," November, 1921, and reprinted here by permission of the author.

THE GENERAL MESSAGE

It is nearly three centuries ago that Sir Thomas Browne wrote, for private circulation among a few of his intimate friends, his *Religio Medici.* He did not call it *Religio Medicorum,* for he could not assume to speak for others. Indeed, as he confessed, the men who practiced the healing art in his time were generally, like loud-mouthed boys in their teens, boastful of their atheism; but for himself, a humble practitioner in an acient town of England, he could testify; and he, the giver of balm for the body, could unobtrusively pass the message to others, as a balm for the spirit.

And so, to-day, I speak not of the *Religio Mathematicorum,* for I have not the authority; I feel, indeed, that *Religio Mathematici* is a misnomer; but I also feel that it is proper to speak of the relation of mathematics to a religious attitude of mind, and I know of no better title for my purpose than the one I have chosen.

Do not, however, feel that the message seeks to change the faith or lack of faith of any man; do not feel that it contains a plea for any creed or for any sect; do not feel that it sets forth anything that is new; but rather feel that it represents the mere commonplace knowledge that most of us have, and the mere basis of belief that mathematics may possibly foster. It goes no farther, it seeks only a single corner of the foundation walls upon which we may build, it lacks warmth, it lacks beauty, it lacks the fervor of religious faith, and it is hard of texture; but the foundation stones of palaces, and of temples, and of homes all lie below the surface; they, too, are cold, but they sustain structures which give to the world pleasure and protection and repose. The message is transmitted by one who has joined in serious conference with those who hold the Brahmin faith; by one who has learned much from the high priests in the temples of Buddha, who has sat at the feet of the followers of Zoroaster, and who has communed with those who wear the green turban that marks the seed of the Prophet,—one who believes that

"Creed and rite perchance may differ,
yet our faith and hope be one,"

and who, with the author of the *Religio Medici* itself, is able to say:

"For my Religion, though there be several Circumstances that might perswade the World I have none at all, . . . yet, in despight hereof, I dare without usurpation assume the honourable Stile of a Christian."

Specifically, I wish to consider, even though totally unable to answer satisfactorily, these questions: What bond of concord, if any, is there between mathematical knowledge and religious faith? What influence can an exact, abstract, reputedly frigid science like ours have upon the religious nature of man. What, in fact, is the soul of mathematics, and to what wave lengths must our own souls be tuned to catch its message?

Such, you will say, are the imaginations of a dreamer; not the serious thoughts of a mathematician. So be it. Were it necessary to make the choice, I would rather be a dreamer without mathematics, than a mathematician without dreams or a teacher without imagination. What I wish to show to those who are not of our calling, however, is that there is no other science that leads so directly to a recognition of the reasonableness of a broad religious faith, and none that parallels so completely the broader tenets of the Fathers.

WHAT IS RELIGION?

In the domain of mathematics we find it necessary at certain times and convenient at certain other times to define our terms; but unless we need a particular term in a proof we do not feel bound to define it with precision. For example, in elementary mathematics we do not deem it necessary to define space, number, or straight line, the terms being basic and, for the beginner, indefinable.

So if you ask me to define religion before we consider its relation to mathematics, I must simply refer you to the theologians, wishing you good fortune in your quest. To many people the Buddhist is not religious, but I have often found him intensely and beautifully so. To many religiously-minded persons the Parsee is a heathen, but I have seen as great faith and as pronounced goodness of work among the Parsees as I have found among many Christians with whom I worship. For our purposes, therefore, I prefer to think of religion with respect to certain general characteristics, one of which is faith; faith as a very

intellectual and catholic-minded writer has remarked, in "The Eternal, not ourselves, that makes for righteousness"; faith that this life does not end all; faith that our lives today prepare for our lives somewhere beyond.

TANGIBLE IMMORTALITY

Out of the manifold characteristics of the mind of youth, one of the most interesting is best described by the word "opinionatedness"—a word that I grant should be in the *Index expurgatorius.* All normal individuals reach this milestone, most of them pass it, but some find their development arrested at this point, and here they bind themselves for life. In particular, they absorb (I will not say develop) the idea that immortality is an idle word of an idle faith, a part of the "opiate of the proletariat," as our most modern autocracy has phrased it.

One thing that mathematics early imparts, unless hindered from so doing, is the idea that here, at last, is an immortality that is seemingly tangible,—the immortality of a mathematical law. The student of algebra, for example, may well question the use of the traditional curriculum, but when he finds the value of $(a+b)^2$ he has come in contact with an eternal law. The laws of the Medes and Persians, unchangeable though they were thought to be, have all perished; the canons that bound Egyptian activities for thousands of years exist only in the ancient records, preserved in our museums of antiquity; the laws of Rome, which at one time dominated the legal world, have given place to modern codes; and the laws that we make to-day are certain to be changed to-morrow. But in the midst of all these changes it has ever been true, it is true to-day, it shall be true in all the future of this earth, and it is equally true throughout the universe whether in the algebra of Flatland or in that of the space in which we live, that $(a+b)^2 = a^2 + 2ab + b^2$.

We may change the symbols,—they are temporatry expedients to convey the idea; we may speak in different tongues,—they are local expedients to convey thought; but it is inconceivable to us that the relation which the formula expresses should not be true always and everywhere,—a tangible symbol of the immortality of law.

What I learned in chemistry, as a boy, seemed true at the time, but much of it to-day is known to be false. What I learned of molecular physics seems at the present time like children's stories, interesting but puerile. What we learn in history may be true in some degree, but is certain to be false in many particulars. So we may run the gamut of learning, and nowhere, save in mathematics alone, do we find that which stands as a tangible symbol of the immortality of law, true "yesterday, to-day, and forever."

But does the teacher make this known to the student? Does the student come to feel the significance of this fact,—this fact so full of awe to the normal mind, this evidence of immortality that never comes to his consciousness until he meets it in mathematics? I do not know, nor do I know how much else that is great, that has tremendous significance, is taught or is not taught in science, in letters, in history, or in art. I only know that mathematics can do this thing; that it can (and it should) give, to the degree that the pupil is able to receive it, the idea that before the world was created, before our solar system was formed, and after our system shall cease to be, the everyday laws of mathematics stood and shall stand for immortal truth,—for laws that are divine in their infinite endurance. It is not necessary, it is not desirable, that we should preach these things in the halls of learning; but it is essential that we should feel their significance. This done all the rest "shall be added unto you." Stated in another way, the immortality of law means that we come in touch with the invariant. The tyro in mathematics comes early upon the invariant properties of a figure as seen in the theory of elementary projection. In a wider sense, however, all geometry is a science of invariance. We prove a law for a general plane triangle and it never varies, whatever we do to the figure. If we prove that $a^2 = b^2 + c^2 - 2bc \cos A$, then, however A may change, the law itself will never vary. In it the pupil comes into touch with the unchangeable, with the absolute.

It is the same with all other laws of geometry. In any convex polyhedron, whatever its shape, the law remains that the number of faces plus the number of vertices is equal to the number of edges increased by two.

"Change and decay in all around I see," but the established properties of a general geometric figure, in our space, are as unchangeable in that space as divinity itself. Stated in still another way, the immortality of law and the invariability of mathematical principles mean the eternity of mathematics. To come into relation with a science which was illustrated by the spiral nebulae before our solar system was formed, which only now reveals to us those laws of crystals which were in operation long before life appeared upon the earth, and which is also entirely independent of matter, so that if we could imagine the universe destroyed absolutely, the laws would still be true,—to come into relation with such a science makes real to us, as no other discipline in our curriculum can possibly do, the ideal of truth eternal.

OUR INFINITESIMAL NATURE

I know of nothing which acts as such a powerful antidote to that which I ventured to call "opinionatedness," as a study of mathematics. To know that the light from solar systems far larger than our own has been thousands of years in reaching us, gives us an idea of our infinitesimal nature, in comparison with space about us, that can come only with a study of the science that is is ours to teach. A bacillus in our veins, so small as to be invisible through a powerful microscope, is a giant compared with ourselves in our relation to this space in which we live. Our doubts, our beliefs, our hopes, our fears are all so trivial, so infinitesimal, so like a lost electron in our solar system, as compared with our relative importance in the universe as revealed to us by the calculations which mathematics brings to bear upon the great problem! Cowper wrote well when he put in verse the words,

> "God never meant that man should scale the heavens
> By strides of human wisdom,"

and even the mathematics of youth confirms the thought.

With our feeble voices we cry out that we are certain that this life ends our existence, but mathematics shows us the truth of Voltaire's words, in speaking of human opinion of the signficance of life:—

"Doubt is not a pleasant condition, but certainty is an absurd one."

Our feeble voices join in a protest that we cannot understand God. Again the "old man of Ferney," condemned by so many as an atheist, speaks out and says,

"My reason tells me that God exists; but it also tells me that I cannot know what He is."

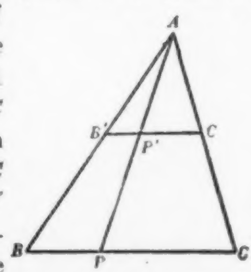
The infinitesimal still cries out that it cannot understand what our soul is, and again the sage speaks:

"Is it indeed likely that we should know what our soul is, when we can form no idea of light if we had the misfortune to be born blind?"

We agree that spaces of higher dimensions than the one in which we think we live can easily be conceived by analogy, and we agree without question to the paradoxes which we meet in the study of infinity, and yet we feel that it shows our great wisdom, or perhaps our boldness, if we deny the soul an existence. Strange, that in algebra we accept without the slightest question the idea of the permanence of law, but that our little natures should so often boast that we deny the permanence of the soul!

CONTACT WITH THE INFINITE

One of the impressive experiences which come to the devotees of our science is the continual contact with the infinite,—an experience which is inspiring beyond words to express,—and sometimes as discouraging. To take an illustration that is a mere commonplace to those who have even visited the borderlands of our science, suppose that a line segment $B' C'$ joins the mid-points of the sides $A B$ and $A C$ of the triangle $A B C$, and that the straight line $AP'P$ connects the vertex A with points P' and P on $B' C'$ and BC respectively. It is then evident that, in this construction, to any point P on BC there is one point and only one point P' on $B'C'$ that corresponds to it, and conversely. Like it or not as we please, there



is only one possible conclusion, namely, that there is a one-to-one correspondence between the points on $B'C'$ and the points on BC; or, taking the usual definition of equality, that there are precisely as many points on $B'C'$ as there are on BC, which is

twice as long. But similarly, the number of points on $B'C'$ can be shown to be the same as the number on any part of BC, and so a part (not of a line, but of a group of points) is equal to the whole. To adopt a paradox of old Tertullian, "Certum est quia impossible est."

Against such a conclusion our little minds revolt; we have been taught, or we have empirically learned, that the part is less than the whole,—as it really is in dealing with the finite. But when we come to deal with the infinite, our little, narrow, finite laws break down; they are even more feeble than the beliefs and experiences of the child, when viewed by the eyes of a mature man. We must face the inevitable, that in the domain of the infinite, the part may indeed be equal to the whole, in spite of the childish beliefs of our finite minds. The mathematician continually meets this necessary conclusion; it is a commonplace; to him the laws of the finite give way without question when he enters the domain of the infinite. In all the finite, indeed, he sees the infinite. He knows that there is a one-to-one correspondence between the points in a sphere and those in the entire universe about him, and between the number of vibrating points in his brain and the number in all space. In other words, within the brain of each of us, there is a point that corresponds to any given point in the universe, say one on the surface of Neptune. If we take another point, say the center of a fixed star, there is one special point, and only one, within each brain that also corresponds to that. If we move the point in the fixed star ever so slightly, we move the point in the brain so that it shall continue to correspond to it. Does all this signify anything to us? I certainly do not know. Does it mean that the planets have their influence upon us? I do not know, but with my finite mind I am led to say I do not think so. Does this mathematico-physical relation of our brains to all the universe about us have any deeper significance? I do not know. But one thing I do know, that thoughts like these give new meaning to the words, "For, behold, the Kingdom of God is within you."

Do such ideas signify that it is ours to preach? Shall mathematics become a medium for religious instruction? Do we not lower religious belief when we link it to the certainties of cold mathematics? With respect to each of these questions I would

reply, "I do not think so." With respect to the last one, I see no difference in the sanctity of truth, whether the truth be taught in the books of Euclid, in the holy books of the East, or in the Christian Church:—in the significance of truth,—yes; in the *sanctity* of truth,—no.

OUR IMPOTENCE IN RELATION TO THE ETERNAL

I know of no other branch of learning that makes so clear to us our impotence in relation to the Eternal. We fail today in a problem in chemistry, but we feel that we may succeed tomorrow; we are at a loss to know how to overcome a certain difficulty in physics, but we have confidence that someone, sometime, somehow, will overcome it; we do not attain to the success we had hoped for in the painting of a certain picture, but we "carry on" in the hope of bettering our work from day to day; and so in mathematics, we fail but we persevere. But there come times in mathematics when we fail and know that we must fail, because we come in conflict with the Eternal. At first sight we say that we can construct a seven-edged polyhedron. We fail, we seek out the reason, and we find that we are combating the everlasting truth of which I have spoken, the truth that $f + v = e + 2$. Protest as we will, we are powerless when we combat the Eternal.

PHYSICAL PERMANENCE

Our mathematics also comes to the aid of science and assists in the proof that force is not lost, and shows us that even a thought generates a wave which has its eternal influence. I like the ancient theory, and I know of no reason why it should not be a fact, that the very walls and arches of the venerable cathedrals of the old world, which have heard the daily chants of the priests for many centuries, have come to vibrate in unison therewith, so that the same singers, if transported to another spot, fail to produce the same sonorous, musical effect. Why, then, should not the thoughts of others influence us, as the telepathists affirm, and why should not our body of thought be permanent in space, even from a materialistic point of view?

Mathematics is such a science of harmony! The universe is such a science of harmony! The "music of the spheres" meant precisely this in the ancient philosophy;—and how out of all har-

mony with what mathematics reveals to us is the theory of annihiliation, even, as I have said, from the materialistic standpoint!

To quote again from Voltaire, whose name is anathema to so many ignorant religionists:

"All nature cries aloud that He *does* exist; that there *is* a supreme intelligence, an immense power, an admirable order, and that everything teaches us our own dependence upon it."

Spoken like a philosopher, but also like the mathematician that he was; for Voltaire was a great student of Newtonian philosophy and did more than any man in his century to make the teachings of Newton known to the general intellectual element of France. That Voltaire, the bitter antagonist of fraud and sham, in the Church as well as in the State, should honestly confess his faith in this manner need not surprise us. It shows his greatness. Cicero told the world two thousand years ago that the greatest thinkers always have such faith:

"There is, I know not how, in the minds of men a certain presage, as it were, of a future existence; and this takes the deepest root and is most discoverable in the greatest geniuses and the most exalted souls."

THE DRAMA OF SPACE

What a science is ours that raises the curtain on the drama of space! That shows us a finite space in which all bodies are at rest until acted upon by some external force; and then the space of the infinitesimal, in which a radically different code of laws obtains,—where everything (molecules, atoms, electrons, and very likely sub-electrons) is an automaton, self-moved and never resting. John Burroughs has expressed this idea, adding:

"When we reach the astronomic world, or the sidereal universe, we find the same condition that prevails in the world of the infinitely little: perpetual motion goes on, friction is abolished, and nothing is at rest. . . . Height and depth, upper and under, east and west, north and south, weight and inertia, as we experience them, have vanished. There are no boundaries, no ending and no beginning, no center and no circumference; the infinite cannot have any of these."

What a science is ours, moreover, that reveals to the youth the secrect of indirect measurement! That shows him how the dis-

tances to the stars are found! That opens to him the moving picture of the universe! That reveals the world of the electronic bodies as the reciprocal of the world of the sidereal bodies! And who can measure the influence of this revelation upon the soul of one who is standing upon the threshold of young manhood or young womanhood?

SCIENTIFIC RELIGION

Religion is generally felt to be unscientific. One of the world's great authorities, Professor Harald Hoffding of Copenhagen, in his *Philosophy of Religion*, speaks of it as that state of mind "in which feeling and need, fear and hope, enthusiasm and surrender play a greater part than do meditation and inquiry, and in which intuition and imagination have the mastery over investigation and reflection."

I cannot believe that this should be the case with respect to the great basal facts of religion, although it is so traditionally, at least in our sectarian doctrines. We lay down certain postulates in geometry; they may, as in the case of parallels, be true or false; mathematics simply says, "If *A*, then *B*,"—if these postulates are sound, then these conclusions are true. I have often wondered mildly why religion did not do the same, postulating certain statements and then proceeding precisely as in mathematics,— "If *A*, then *B*." The postulates, like those of Euclid, might be true or false, but the deductions would be absolute. I have no idea as to what postulates would be assumed; I simply know that it seems entirely scientific to assume at least a few that are reasonable. We do not, in elementary mathematics, feel that our postulates must be independent or that they must cover all possible needs; we simply assume what seems to be necessary for the mind of youth and on that we build.

When I think, in a kind of indefinite fashion, of what my mind postulates with respect to some of the larger features of mathematics, my thought runs to something like this:

1. The Infinite exists.
2. Immortal laws exist.
3. The laws relating to finite magnitudes do not hold respecting the infinitely large or the infinitely small.
4. The existence of hyperspace is entirely reasonable.
5. No factor is ever lost.
6. Time may be a closed curve.

Such a list of postulates might easily be put into theological language as well, and might be extended when necessary. For example, the theologian might phrase these same postulates like this:

1. God exists.
2. God's laws exist.
3. God's laws are entirely different from ours.
4. There are spaces beyond ours.
5. The soul exists and is eternal.
6. God looks at time as a whole.

After making out his list of postulates, the theologian might formulate his definitions. It would be as easy, I should think, to make an attempt to define God, heaven, angel, and miracle as it is to make at attempt to define infinity, hyperspace, straight line, solid angle, and dozens of other terms that we use in mathematics; or, if precise definition were found unnecessary, as is the case with these mathematical terms, at least some reasonable limitations would be in order.

Given the postulates and the definitions, I see no reason why a perfectly rigorous set of propositions should not be erected and religion put on a cold, scientific foundation. I should not wish to see this done; I think it would be about as sensible as to build up a scientific, deductive system of love or beauty; but what I mean to say is that if religion is unscientific, it is partly because the world wishes it to be so. Love would not be any more potent, or more real, or more beautiful if we formulated a set of postulates and deduced a series of propostions relating to it; and the same may be said of religion.

DUALITY OF MATHEMATICS AND RELIGION

Schopenhauer's duality between Time and Space is well known. Time is homogeneous, for example: it is a continuum, and no part is separated from any other part by something which is not time. The dual proposition for Euclidean space is simply formed by a substitution of "space" for "time."

We need not be surprised at this dualism. Time has often been called a fourth dimension, and it harmonizes with our three-dimensional Euclidean space to think of time alone as a space of one dimension. The two united gave to Sir William Rowan Hamilton the science of quaternions, and now they unite again to give birth to one feature of the Minkowski-Einstein hypothesis.

How much of Schopenhauer's duality is real we cannot say, because we cannot say how much of time and space are real. If we leave the domain of Euclidean space and think of ourselves as living in a space that curves through a fourth dimension, as a spherical two-dimensional space curves through a third dimension, then this space becomes limited, as Einstein suggests, and we are led to consider other three-dimensional spaces like ours, —all quasi-finite, and the Schopenhauer dualism automatically changes.

Similarly, when we consider the duality between mathematics and religion, we have nothing positive. Change the space in which we live, make time a fourth dimension, let our new universe curve through a fifth dimension, and the details of the parallelism would necessarily change. Grant, however, certain postulates, derived empirically like the postulates of mathematics, or, as we say, derived from common sense,—grant these, and some such parallelism as the following is suggested to the mathematical mind:

Mathematics	Religion
1. The Infinite exists.	1. God exists.
2. Eternal laws exist.	2. Eternal laws exist.
3. The laws relating to finite magnitudes do not hold respecting the infinitely large or the infinitely small.	3. God's laws are so different from ours as to be absolutely non-understandable by us.
4. The existence of hyperspace is entirely reasonable.	4. The existence of a heaven, with gradations, is entirely reasonable.
5. No factor is ever lost.	5. The soul is eternal.
6. Time may be a closed curve.	6. God looks at time as a whole.
7. Time may be a fourth dimension.	7. In the next world, the direction of time may actually be seen.
8. Positive infinity may physically coincide with negative infinity, if lines curve through fourspace.	8. In God's sight the infinite past and the infinite future are the same.
9. A Flatlander has enough of the third dimension in his being to give him some feeling of that dimension; and so this may explain the fact that we have some feeling of the fourth dimension.	9. The human soul has enough of the divine within it to have some feeling of the reality of divinity and of the world beyond.
10. Mathematics is a vast storehouse of the discoveries of the human intellect. We cannot afford to discard this material.	10. Religion is a vast storehouse of the discoveries of the human spirit. We cannot afford to discard this material.
11. It is not necessary that the solution of a problem, by limited means,—say the trisection of an angle,—should be found in order that we may feel certain that the problem can be solved by *some* means.	11. It is not necessary that the solution of the problem of religion, by our limited human means, should be found in order that we may feel certain that the problem can be solved by *some* means.

12. Every term in an infinite sequence is in a small way a part of infinity.	12. Lucretius spoke wisely when he said, "Everyone is in a small way the image of God."

CONCLUSION

And what is the conclusion? Does mathematics make a man religious? Does it give him a basis for ethics? Will the individual love his fellow man more certainly because of the square on the hypotenuse? Such questions are trivial; they are food for the youthful praragrapher. Mathematics makes no such claim. What we may safely assert, however, is this,—that mathematics increases the faith of a man who has faith; that shows him his finite nature with respect to the Infinite; that it puts him in touch with immortality in the form of mathematical laws that are eternal; and that it shows him the futility of setting up his childish arrogance of disbelief in that which he cannot see.

And if this be the case, then what is the duty of teachers of our science? To preach?—that should be the last thought. The greatest sermons are preached in silence. The most ancient religions that we have, if there be more than one fundamental religion, have always recognized this fact. And so it must be with us,—that we should teach "the science venerable" not merely for its technique; not solely for this little group of laws or that; not only for a body of unrelated propostions or for some examination set by the schools; but that we should teach it primarily for the beauty of the discipline, for "the music of the spheres," and for the faith that it gives in truth, in eternal law, in the Infinite, and in the reality of the imaginary; and for the feeling of humility that results from our comparison of the laws within our reach and those which obtain in the transfinite domain. With such a spirit to guide us, what teachers we would be!—whether of those who are standing on the threshold, of those who are passing through the realms of mystery that lead to manhood and womanhood, of those of mature years, or of those who, as the ancients were wont to say, "number their years upon their right hand." For then, unconsciously but none the less surely, would we prove the words of a seer among poets:

> "And Reason now, through number, time, and space
> Darts the keen lustre of her serious eye;
> And learns, from facts compared, the laws to trace
> Whose long procession leads to Deity."

AN ELEMENTARY EXPOSITION OF THE THEOREM OF BERNOULLI WITH APPLICATIONS TO STATISTICS

By Professor H. L. RIETZ, University of Iowa

In connection with the wide use of certain statistical methods within the past few years, there is coming to be some recognition of the importance of establishing a measure of the degrees of confidence that can properly be placed in inferences from statistical results such as mean values, standard deviations and coefficients of correlation. This recognition is shown in the increased application of probable errors, even if the applications are in many cases made without knowledge of the derivations or limitations of the formulas employed. The need for such criteria in passing judgment on the significance of the simplest of statistical results may perhaps be made clear by an appropriate illustration.

In the third edition of American Men of Science by Cattell and Brimhall, it is stated that a group of "scientific men reported 716 sons and 668 daughters." The valid inference is drawn that the "difference falls within the limits of chance variation, and is not likely to be significant." On the same page, 804, we find that a group of scientific men report 1705 brothers and 1527 sisters. These data suggest the following questions of simple statistical sampling: What is the probability in throwing 1384 coins that the number of heads will differ from $\frac{1384}{2} = 692$ by as much or more than $692 - 668 = 24$? What is the probability in throwing 3232 coins that the number of heads will differ from $\frac{3232}{2} = 1616$ by as much or more than $1705 - 1616 = 89$? The answers to these questions are obtained by an application of what is known in certain important mathematical literature* as the theorem of Bernoulli, although the theorem in the form in which we shall use it contains much in addition ** to the Bernoulli theorem as it appeared in the latter's works.

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* See German and French Encyclopedias of Mathematics—Papers on Probability.

** See Laplace, Theorie Analytique des Probabilities, Introduction, p. XLVII, Bertrand, Calcul des Probabilities, Chapter V.

THEOREM OF BERNOULLI

We shall assume that during a set of s trials, the probability of the happening of an event is a constant p from trial to trial, and that

$$p + q = 1.$$

Then the probabilities that the event will happen exactly s, $s - 1$, $s - 2$, $\cdot \cdot \cdot$, 1, 0 times in s trials are given by the successive terms of the *binomial* expansion

$$(p+q)^s = p^s + sp^{s-1}q + \qquad + \frac{s!}{m!(s-m)!}p^m q^{s-m} + \dots + spq^{s-1} + q^s, \tag{1}$$

To find the "most probable" number of happenings, we seek the value of m to which a maximum term of (1) corresponds. It is easly shown that

$$m = ps$$

gives the maximum term if ps is an integer. When ps is not an integer, the integer m is such that

$$ps - q \gtreqless m \gtreqless ps + p$$

gives a most probable value.

When $ps - q$ and $ps + p$ are integers, there occur two equal terms in (1) each of which is larger than any other term of (1). For example, note the equality of the first and second terms of the expansion of

$$(\tfrac{5}{6} + \tfrac{1}{6})^5.$$

For the present, let us assume that ps is an integer, and let us represent the terms of (1) by ordinates of the curve

$$y_x = f(x),$$

where x marks deviations from the maximum term as an origin. Then we have

$$y_x = \frac{s!}{(ps-x)!(qs+x)!}p^{ps+x}q^{qs-x}, \quad . \quad . \quad . \tag{2}$$

and

$$y_{-x} = \frac{s!}{(ps-x)!\,(qs+x)!}p^{ps-x}q^{qs+x}, \tag{3}$$

By making $x=0$ in (2) or in (3), we have the maximum ordinate

$$y_o=\frac{s!}{ps!\,qs!}\,p^{ps}q^{qs}.$$

With values of s that are reasonably large for statistical purposes, it is usually impractical to calculate the factorial in (2) and (3) without some special methods of approximation. Such a method is provided by an application of Stirling's theorem for the representation of large factorials.
This theorem states that*

$$s!=s^{s+\frac{1}{2}}e^{-s}\sqrt{2\pi}\text{ approximately.} \tag{4}$$

The substitution of this value for $s!$ and corresponding values for $ps!$ and $qs!$ in

$$y_o=\frac{s!}{ps!\,qs!}\,p^{ps}q^{qs}$$

gives, after some simplification,

$$y_o=\frac{1}{\sqrt{2\pi spq}},$$

To illustrate, the most probable value in throwing 1000 coins, namely 500 heads and 500 tails, has a probability

$$y_o=\frac{1}{\sqrt{500\pi}}=.02523.$$

It is important to note that this most probable value is not likely to be obtained in a single trial since its probability is only a little more than $\frac{1}{40}$. It may be of interest to the reader to compare the simplicity of the calcuation of y_o as above with the calculation of

$$\frac{1000!}{500!\,500!}\left(\frac{1}{2}\right)^{1000}\text{ by logarithms.}$$

By the application of Stirling's theorem to (2), we obtain, after slight simplification,

$$y_x=\frac{1}{\sqrt{2\pi spq}}\left(1+\frac{x}{ps}\right)^{-ps-x-\frac{1}{2}}\left(1-\frac{x}{qs}\right)^{-qs+x-\frac{1}{2}} \tag{5}$$

* For proof, see Whittaker and Watson, Modern Analysis, Third Edition, p. 253; Czuber, Wahrschenlichkeitsrechnung, I, 1908, p. 22.

In practical statistical problems, we are in general interested in deviations, x, that are small compared to the most probable value ps. In fact, we may well confine our attention to deviations that are of the order of $\sqrt{s}$ when we are discussing fluctuations in sampling. With this limitation on x, it is shown in the appendix to this paper that the sum

$$y_x + y_{-x} = \frac{2}{\sqrt{2\pi pqs}}\, e^{-\frac{x^2}{2pqs}} \text{ approximately.} \tag{6}$$

Since we very commonly make our inquiries about a given deviation on either side of the most probable, we are interested in the sum $y_x + y_{-x}$ given by (6).

The limitation that ps be an integer may well be removed. From what was shown above about the most probable value, we may in any case write for the most probable number of happenings

$$m = ps + k$$

where $-1 < k < +1$.

With larger values of s, it can be shown that the difference brought about by the use of $ps + k$ instead of ps is of negligible importance in our problem of fluctuations in sampling.

We are particularly interested in finding the probability that the variable deviation x will remain within assigned bounds, say within d and $-d$ inclusive. To find this probability requires a method of finding the sum

$$y_d + y_{d-1} + \cdots + y_1 + y_0 + y_{-1} + \cdots + y_{-d} = \sum_{x=-d}^{x=d} y_x. \tag{7}$$

As there is likely to be a large number of y's, some special method of finding the sum is of practical value. The sum of such a large set of ordinates may be found by the Euler-Maclaurin formula of the calculus of finite differences. For the purpose of a sampling problem, a simpler method of approximation than that provided by the Euler-Maclaurin Theorem is suitable. It will be convenient in what follows to make

$$y'_x = \frac{y_x + y_{-x}}{2} = \frac{1}{\sqrt{2\pi pqs}}\, e^{-\frac{x^2}{2pqs}}. \tag{8}$$

We may now well conceive of obtaining the approximate sum of the ordinates in (8) by finding the area enclosed by the curve, the x-axis, and the ordinates $x = -d - \frac{1}{2}$, and $x = d + \frac{1}{2}$.

For this purpose, we use as an approximate value of the area, the integral,

$$\int_{-d-\frac{1}{2}}^{d+\frac{1}{2}} y'_x dx = 2 \int_{0}^{d+\frac{1}{2}} y'_x dx$$

$$= \frac{2}{\sqrt{2\pi pqs}} \int_{0}^{d+\frac{1}{2}} e^{-\frac{x^2}{2pqs}} dx \tag{9}$$

The theorem of Bernoulli may now be stated by saying, (1) *that ps is a most probable value,* (2) *that formula* (9) *gives the probability that a deviation x from the most probable will not exceed an assigned deviation d on either side of the most probable.*

The numerical value of (9) is readily obtained in any particular case by the use of the normal probability integral as we shall show by applications to the numerical questions proposed on pp. 1 and 2.

In the first question,

$$s = 1384,$$
$$pqs = 346,$$
$$\sqrt{pqs} = 18.601,$$
$$d = 716 - 692 = 24,$$
$$\frac{d + \frac{1}{2}}{\sqrt{pqs}} = 1.3171.$$

From a table of the normal probability integral (Table IV, Davenport, Statistical Methods), we find for the deviation

$\frac{x}{\sigma} = \frac{d + \frac{1}{2}}{\sqrt{pqs}} = 1.3171$, the value of (9) to be $P = .8122$.

This is the probability in throwing 1384 coins that the deviation of the number of heads from $\frac{1384}{2} = 692$ will not exceed 24. The probability of a deviation greater than 24 is then $1 - .8122 = .1878$. Expressed in another way, we may say we should predict that, in the long run, a deviation greater than 24 on either side of the most probable will occur slightly less than once per five trials. In the second question,

$$s = 3232,$$
$$\sqrt{pqs} = 28.425,$$
$$d = 1705 - 1616 = 89,$$
$$\frac{d + \frac{1}{2}}{\sqrt{pqs}} = 3.1486.$$

Referring now to a table of the normal probability integral, we find for the deviation

$$\frac{x}{\sigma} = \frac{d + \frac{1}{2}}{\sqrt{spq}} = 3.1486,$$

the value of (9) to be

$$P = .99836.$$

The probability of a deviation greater than the assigned deviation on either side of the most probable $\frac{3232}{2} = 1616$, is then $1 - .99836 = .00164$.

In other words, we predict a deviation larger than 89 on either side of the most probable should occur in the long run about once per $\frac{1}{.00164}$ trials, or roughly once per 600 trials. We thus have a quantitative criterion to judge of the significance of the given deviation compared to fluctuations in simple sampling.

By dividing the frequencies under consideration by the total numbers involved, we have relative frequencies, and the theorem of Bernoulli may be stated in terms of these relative frequencies. We should then regard the theorem as furnishing a criterion for testing whether the deviation of a statistical ratio from an assumed probability can be reasonably regarded as a fluctuation in sampling.

To summarize, we may say that the theorem of Bernoulli states (1) the number of happenings that is most probable, (2) the probability, in making s trials with constant probability p, that

the departure of the number of successes from sp will not exceed a given number, or that the departure of a statistical ratio from a known constant probability p will not exceed a given value.

The converse theorem to that of Bernoulli is of great importance in statistics. That is, to determine the probability that an unknown probability of an event will not deviate more than an assigned value from a statistical ratio is a problem of much interest. We shall not attempt here to give the reasoning by which the converse is established because of limitations of space and because the purpose of this paper is accomplished by giving a view of the method of treating one of the simplest problems of fluctuations in sampling.

Appendix

Given the function

$$y_x = \frac{1}{\sqrt{2\pi spq}}\left(1+\frac{x}{ps}\right)^{-ps-x-\frac{1}{2}}\left(1-\frac{x}{qs}\right)^{-qs+x-\frac{1}{2}} \qquad (1')$$

marked (5) above, to show that

$$y_x + y_{-x} = \frac{2}{\sqrt{2\pi pqs}}\, e^{-\frac{x^2}{2pqs}} \quad \text{approximately} \qquad (2')$$

under the limitations on x stated on p. 5.

From (1′)

$$\log\ y_x = -\tfrac{1}{2}\log\ (2\pi pqs) - (ps+x+\tfrac{1}{2})\log\ (1+\tfrac{x}{ps})$$
$$-(qs-x+\tfrac{1}{2})\log\ (1-\tfrac{x}{qs}).$$

By expanding $\log\ (1+\frac{x}{ps})$ and $\log\ (1-\frac{x}{qs})$ in series, and simplifying, we have

$$\log y_x = -\tfrac{1}{2}\log.\ (2\pi pqs) + \frac{(p-q)x}{2pqs} - \frac{x^2}{2pqs}$$
$$+\frac{x^3}{6p^2s^2} - \frac{x^3}{6q^2s^2} + \cdots. \qquad (3')$$

Hence,

$$y_x = \frac{1}{\sqrt{2\pi spq}}\, e^{\frac{(p-q)x}{2pqs} - \frac{x^2}{2pqs} + \frac{x^3}{6p^2s^2} - \frac{x^3}{6q^2s^2} + \cdots}$$

$$=\frac{1}{\sqrt{2\pi spq}}\,e^{-\frac{x^2}{2pqs}}\left(1+\frac{(p-q)x}{2pqs}+\frac{x^3}{6q^2s^2}-\frac{x^3}{6p^2s^2}+\cdots\right) \tag{4'}$$

Similarly,

$$y_{-x}=\frac{1}{\sqrt{2\pi spq}}\,e^{-\frac{x^2}{2pqs}}\left(1-\frac{(p-q)x}{2pqs}-\frac{x^3}{6p^2s^2}+\frac{x^3}{6q^2s^2}\cdots\right) \tag{5'}$$

In the applications to statistics, we are generally interested in deviations, x, that are small compared to ps and qs. In fact, we may well confine our attention in the treatment of fluctuations in sampling to values of x not exceeding $\sqrt{s}$ in order of magnitude. Thus, in (4′) we have not retained in the parenthesis the term $\frac{(p-q)^2x^2}{2(2pqs)^2}$. Under this limitation on x, the sum of terms beyond those written in (4′) and (5′) may be taken as negligible for our purposes.

Hence, we have from (4′) and (5′),

$$y_x+y_{-x}=\frac{2}{\sqrt{2\pi pqs}}\,e^{-\frac{x^2}{2pqs}} \text{ approximately.}$$

THE RELATIVE EMPHASIS UPON MECHANICAL SKILL AND APPLICATIONS OF ELEMENTARY MATHEMATICS

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The teaching of elementary mathematics has two distinct purposes, quite closely connected with each other and yet easily distinguished. The pupil must develop a skill in the manipulation of the symbols of elementary mathematics and he must develop a power to apply them to the solution of life problems. By special emphasis upon either phase of the subject an ability can be developed in it without a corresponding ability in the other phase, and sometimes at the expense of ability in the other phase. It is easy to develop in pupils a high degree of skill in the mechanical processes of computation with almost no ability to decide in a particular problem what process is to be used. And in pioneer days many men were quite able to solve the problems met in daily life though knowing nothing of the use of figures for purposes of computation. It has been very interesting to me to note that pupils in the third and fourth grades, utterly innocent of the use of the symbols in division of fractions, will frequently solve problems involving division of fractions more readily than will those same pupils after they have been taught to "invert the divisor and multiply." This does not mean that skill in computation with figures stands in the way of solving concrete problems, but that during the time devoted to acquiring the skill they have lost power in thinking number relations, possibly through a transfer of interest.

In selecting the content of elementary mathematics shall we be controlled by the purpose of training in ability to solve applied problems or by the purpose of developing a high degree of skill with symbols, or by a combination of the two? If we select the content on the basis of the applications, we should, of course, include only those mechanical processes which are of use in solving problems. In arithmetic, though there are several perfectly good ways of subtracting with figures, if our purpose is purely to give a useful tool to be used in problem solving, we should teach but one of them. If our primary purpose is to

develop skill in the use of symbols in computation, we shall teach him several ways of subtracting and develop skill in each. That we are not agreed on either of these purposes is shown by the fact that both of these plans are being followed in different schools to-day. In algebra if our primary purpose is the ability to solve problems we shall include in factoring only those types which the pupil may need in solving problems. Shall we teach the factors of $a^3 - b^3$? At this moment I can think of no concrete problem whose solution depends either directly or indirectly upon skill in factoring $a^3 - b^3$. If selecting the content of factoring on the basis of its applications, I would then omit this type. If, however, I were selecting the content of factoring on the basis of the mechanical processes I would include this type, for it furnishes a very interesting part of the systematic development of the topic of factoring. Having factored a^2-b^2, it is natural to want to go on to a^n-b^n. In algebra we are constantly facing the question, "Are we teaching a science of algebraic symbols, or are we developing power to use algebraic symbols to solve life problems?" The answer to this question must necessarily be, "We are doing both." But in selecting the content of algebra and arithmetic I shall assume that our *primary* purpose should be the second—to develop power to use symbols in *solving problems.*

If you agree with me in this as the dominating purpose when selecting the content of arithmetic, I believe we must modify the content of arithmetic largely. The change should come not so much in the *kind* of material now given as in the *quantity.* I do not believ that we are teaching mechanical processes not needed in the solution of problems, but I do believe that we are failing to develop an ability to solve problems requiring the degree of skilled secured in computation, and that we are even tending to make problem solving itself mechanical. Pupils are taught to solve problems by types just as they are taught factoring by types. They learn to recognize the process to be performed in a problem, not by thinking the conditions stated, but by recognizing set phrases. Give the problem "One ship can cross the Atlantic in 8 days. How long will it take 3 ships?" and you will probably get the answer "24 days," because the phrasing of the problem suggests multiplication.

As early as the second grade the pupil begins to develop skill in computation with figures. He is also given some experience in thinking number relations. But the greater part of his time and energy is devoted to securing skill with figures and the solving of problems seems to be used merely to give meaning to the mechanical processes. Generally there is not enough experience in problem solving to be successful even in this. It seems to be desirable here to reverse the emphasis entirely. Instead of using problems to illustrate mechanical processes, develop skill in the processes only so far and so fast as it is needed in solving problems. This does not imply that skill in computation is to be developed only through problem solving. Drill is necessary and much of it should be entirely detached from the applications—should be purely manipulation of symbols. But this drill should usually be preceded by applied problems making clear the nature of the process needed and showing the need for skill in it. If this approach through applied problems seems impossible, then certainly after a degree of skill has been secured the processes should be followed rather promptly by applications. To each the manipulation of complicated complex fractions in elementary algebra because the pupil will need that skill in simplifying the results of differentitation in calculus is certainly a waste of energy. I am inclined to make the extent of the mechanical processes entirely dependent upon the need for them in solving the problems which the pupil is soon to be asked to solve. In arithmetic we very properly teach the process of short division after the pupil has had considerable experience with problems in which fractional parts are taken and one group is measured by another. His mastery of this process with larger numbers is followed by problems in which his newly acquired skill is exercised. Our only error here seems to be in developing too little power in problem interpretation and judgment of processes to be used in proportion to the skill developed in computation.

If we are in error at all in the relative emphasis upon processes and applications in arithemetic, it seems to me we are much more so in algebra. In arithmetic we give no processes which are not needed in the applications which follow rather promptly. Our error is purely an error of relative emphasis. But in alge-

bra we give skill in mechanical manipulation of symbols which is not only not needed in the accompanying applications but in some cases is not needed in any future applications. And in most cases the processes are taught before the problems in which they are to be used are faced or understood by the pupil at all.

The introduction to algebraic notation will illustrate this point. In arithmetic, after the pupil has had considerable experience with five objects of various kinds he is given the figure 5, which then has some content for him. On the first page of his algebra he is usually told that in algebra numbers are also represented by letters. The distinction between the symbol *a* and the symbol 2 as representing numbers can have no meaning for him. He should first have some general number experiences, then be given the general number symbol. Let him see that a rectangle 2″ x 3″ contains 2x3 sq. in., then enough similiar cases to conclude that the area of any rectangle is equal to the product of its base and its altitude, and he is soon ready for a symbol which represents the base of *any* rectangle. Through many applications dealing with general number and the building up of many formulas the pupil will have built up a symbolization full of meaning, and will have some knowledge of how to use it from the very applications from which it was derived. In using these general number symbols to generalize and formulate the rules of arithmetic he will necessarily learn how to indicate the fundamental processes. For example, if he works out the rule for the area of a trapezoid and attempts to state it as a formula, he will see that he must have a way of indicating that b and b' are to be added, the result multiplied by h and the product divided by 2. Having had such experiences as this in using symbols to indicate processes which a problem requires him to perform, he will later have little difficulty in learning the order of operations and the use of parenthesis in expressions to be evaluated or interpreted. This approach to the symbolization and mechanical processes of algebra through problem conditions differs widely in both plan and results from the more usual plan of first introducing the symbols, defining their use, and then using examples and, finally, problems to illustrate their meaning and use. The first plan begins with problems, goes to mechanical processes as tools for solving them, and ends with problems in which the

new tool is needed. The second plan begins with symbols and and mechanical processes, goes to problems which illustrate these processes, and ends with mechanical processes for which these problems have prepared. By the first plan algebra is a tool for use in life situations; by the second, a science of number symbols illustrated by concrete conditions. I know, of course, that algebra is essentially both a science of symbols and a tool applied to problem solving, but in selecting its content I believ that the latter conception of it should control, and that only those things should be taught in algebra which will be needed rather promptly to solve problems which may possibly be met with in later life.

This should modify our teaching of algebra in two ways. It would compel us to extend the period of exposure to elementary algebra and to omit many of the processes which we are now teaching. To build up a content for the algebraic notation through formulating the rules of arithmetic it becomes advisable to introduce the new notation soon after the rules have been made by the pupils. This could certainly be begun to advantage as early as the seventh grade, probably a year earlier. In my judgment this early introduction is an advantage to the work in arithmetic as well as an introduction to the symbols of general number.

I feel very confident that if the content of elementary algebra were based entirely on the applications to probable life problems, many eliminations of topics would result, possibly a few new ones would be introduced, and certainly the relative emphasis upon topics would be changed. A great part of the space now devoted to the mechanical processes is given to long division, factoring, complicated fractions, theory of exponents, and radicals. I very much doubt whether any of the texts give problems which involve long division either directly or indirectly. The emphasis upon factoring is out of all proportion to its use in solving problems and many types are given which are not involved directly or indirectly in the solution of the book problems. Types of fractions are taught which the pupil meets in no problems earlier than the calculus, and the same can be said of the work with exponents. By reducing largely the amount of time devoted to these mechanical processes, the processes retained could be given a richer meaning through more abundant applications.

Do the students now completing elementary algebra have real power in the use of algebraic symbols, or have they merely developed skill in recognizing types? We teach a pupil to develop the quadratic formula—that is, we develop it for him and then say "Do you see?" and after a while he does. Suppose we then say to him, "Develop a formula for the linear equation in one unknown." Can he do it? Can he even take the first step and write a general equation in one unknown? He can repeat glibly that "the product of the sum and the difference of two numbers is equal to the difference of their squares," but suppose you ask him to show you that the sum of the sum and the difference of two numbers is equal to twice the greater number, will he use general number symbols and do it? Or give him the little number trick, "Think any number, multiply it by 6, add 24, divide by 3, add 4, divide by 2, subtract the original number. The result is 6." Ask him to show how it works. Can he discover? And yet it is in just these ways of formulating number laws and interpreting general number conditions that the symbolization of algebra is useful. We have not been teaching him to make formulas and state laws, but only to solve a very limited number of type problems. Later in life the problems he meets do not come under the types he was taught to solve, he was not given enough experience in problem solving to develop power in applying the algebraic notation to the problems he does meet, and he joins the ranks of those who insist that algebra is not useful except as mental discipline. It seems to me that we have done enough and probably too much toward developing skill in the mechanical processes of algebra, almost enough toward skill in solving problems whose conditions are definitely set up and according to type, but far too little toward developing power in using the algebraic notation to formulate number laws and set up problem conditions. And I am very confident that this change in emphasis would rather increase than diminish the pupil's ability to master the more advanced mathematics if he reaches it.

In the discussion of algebra I have included under the head of applications both the concrete problem situations to which the processes are to be applied, and the method of thought by which the application to to be made. Using algebraic symbols

to state a number law or a rule I have classed under the applications of algebra. It is in essence the method of algebraic thought. I have endeavored to show that in algebra it is this method of thought that we have failed to teach among the applications. In geometry the same division into mechanical processes, method of thought, and concrete applications exists but the present relative emphasis upon them is quite different from that in algebra. The mechanical processes in geometry are very meager, soon mastered, and therefore not over-emphasized. We mechanize the order of a proof, the definitions, and the theorems in order. But these are necessarily repeated so often that after the first few weeks little emphasis is placed upon the mechanical processes. The contest for emphasis is now between the method of thought in geometry and the applications. When teaching the geometry of similiar triangles, which should we stress most, the method of proof of the theorems or their application to concrete problems? Originally the proofs of theorems were the only applications considered as belonging to geometry. The application of the facts of geometry to life problems was merely mensuration and was degraded to a mere topic of arithmetic. Later it became customary to introduce an increasing number of applied problems immediately after the proof of a theorem, though the proof of the theorem and the problem were kept quite distinct. Recently there has been a marked tendency to introduce as early as the seventh grade the notions and the names practically all of the geometric figures, to teach the theorems of geometry without proof, and to use these facts in a great variety of applied problems. The pupil becomes familiar with the language of geometry. He is led to discover the facts and learns to make accurate geometric drawings. He collects and remembers these facts as a means to solving numerous problems of construction and mensuration. Problems of this kind are found much more within the range of his interests and his abilities than are the problems of business usually given in the seventh grade.

The plan of having inventional, or problem solving geometery, precede demonstrative geometry has several advantages. In algebra the power to use algebraic symbols in formulating general number laws is so directly connected with ability to solve con-

crete problems that each is directly helpful to the other and they should be trained together. In geometry, however, the ability to prove a theorem has little if any connection with ability to understand or to apply it, so that either can be taught without the other. I have sometimes felt that the attempt to combine the two was a detriment to both. The type of mind which is interested in working out the logical demonstration of a theorem is not apt at that time to be interested in using that theorem to measure the height of a tree. And if one is inerested in finding the height of a tree and knows the theorem by which to do it, he is apt to care little for the proof of the theorem. I am very skeptical of the theory that demonstrative geometry can be motivated by applied problems. I much prefer to motivate demonstrative geometry by an easy original exercise than to use a problem in mensuration. If the applications are the interesting things, why have the proofs, is a very natural question, especially in view of the fact that the proofs throw little or no light upon the applications. I believe thoroughly that the method of geometric proof should be retained, probably as a requirement for all pupils, because of its influence upon all our thinking. But I would have a course in geometry preliminary to demonstrative geometry. This course, distributed through the seventh and eighth grades, would include notions of all the geometric forms, skill in constructing them, the theorems about them, and problems applying these facts. This would of course, include the elementary facts of trigonometry along with other facts of similar triangles. Following this I would give a course in demonstrative geometry whose purpose would be the mastery of the method of geometric proof through a well selected sequence of easily proved theorems. Since pupil motive in any school activity is in the success he achieves in it, rather than in any realization of future need for it, I should hope to so present the early theorems and their proofs that the pupils could master them. In that case he will be ready for the next proof, and would regard an applied problem as an obstacle rather than as an indncement to the next demonstration.

By way of summary, then, permit me to say that in arithmetic it seems to me that we have selected the content as we should on the basis of the applications. But I think that we are over-

stressing the mechanical processes in relation to the applications, and are presenting them too early—before the child is prepared for them through the applications. In algebra we have not selected the content on the basis of the applications, as we should, but rather on the basis of the mechanical processes. Much of the work upon the mechanical processes should be eliminated. Much more emphasis should be placed upon the applications, especially the algebraic method of using general numbers. The work in algebra, especially its applications, should be distributed over a longer period. In geometry I believe that the stress is where it should be, upon the applications, but that the applied problems of geometry may well precede and be entirely separated from the training in the geometric method of proof.

MATHEMATICS AS FOUND IN SOCIETY: WITH CURRICULUM PROPOSALS

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The curriculum of the future must find its justification in the needs existing in society. Those who will determine the curriculum will be men of extensive vision who can feel the pulse of social needs. The so-called subject-matter specialist, by the very nature of his specialization, is not in a position to settle the place of his subject in the curriculum—but once the sociologist determines that any subject should be found in the curriculum and specifies the objectives that should govern it, then it is the place of the subject-matter specialist to arrange the subject matter and the methods of teaching.

But it is not possible to start with a clean slate in curriculum making. We already have a working curriculum, more or less founded on social needs, more or less founded on tradition. We cannot scrap the curriculum—we must change it to more truly satisfy social objectives.

It is the purpose of this paper to state the uses of mathematics as found in society and hence to make curriculum proposals. But as stated above, such proposals must wait for relative valuations between different subjects by wide visioned sociologists before any recommendations by subject-matter specialists can be considered as final.

The population figures below (intended to cover territorial United States) are the roughest of estimates for purposes of comparison only.

1. Mathematical Research

There are a number of individuals, both men and women, who are on the frontier of the science of mathematics, conducting research and investigation. Their number may be estimated at 8,000. They are probably nearly all found in some institution of higher learning, either as instructors or as advanced students. Some few may also be research experts in science or industry

who further mathematical knowledge in the course of their work. To these is entrusted the most essential burden of carrying on and furthering the great body of mathematical knowledge, so much of which has been used as the very foundation of our civilization. These people are very superior in intellect, although many of them excel predominantly in capacities or powers of a special analytic nature.

II. Mathematics Applied

A second group consist, of those in essentially mathematical occupation, such as inventors, engineers, statisticians, surveyors, scientific men, astronomers, accountants, and research workers in manufacturing, industry, construction, and architecture, who intelligently use and apply the already developed mathematics, but do not necessarily strike out into new mathematical fields. In this group I place all those who make use of mathematical principles in different occupations essentially mathematical in character. These all are high in general intelligence, and especially in analytic ability. They are essential in our modern specialized society. Their number is placed at 80,000.

III. Computing and Drafting

A third group consists of those who perform more or less mechanical mathematical tasks, as computers, cost computers, draftsmen, and bookkeepers. Such are found in banks, offices, and laboratories, wherever computing, drafting or drawing is done. In this group I place those who calculate, compute, read from tables and make tables, use calculating machines and instruments, and in general carry out the directions which have at some time or other been worked out by such as are in group two. These have medium to high intelligence, but have good mechanical ability in the processes of computation, tabulation, drawing, etc. Their number is estimated at 300,000.

IV. Mathematical in Non-Mathematical Occupations

A host of other specialized non-mathematical trades and occupations use mathematics to a very limited extent—the arithmetical processes, formulae, graphs, mensuration, computing or measuring instruments, etc. These uses are in the main so specialized that they may be taught by rule or practice with

little emphasis on principle or generalized procedure. These workers are of all grades of intelligence and we will estimate their number at 20,000,000.

V. Mathematics for Consumers and Users

This group is nearly universal in extent. It consists of those who use mathematics as consumers and users—for buying, investing, general reading, personal accounts, reading meters, clocks, time-tables, making change, etc.

V. Mathematics for Forming Social Judgments

Mathematics is used by many in forming social judgments. This group is steadily becoming larger but at present comprises those of superior intelligence who read and form judgments on facts . Such people are able critically to interpret statistical data through the use of graphs and tables and percentages, and by means of the concepts of the various measures of central tendency, variability, correlation, and reliability. The number in this group may be estimated at 20,000.

VII. Mathematics for Forming Business Judgments

In this group are those who think in terms of interest, discount, sinking fund, annuity, amortization, depreciation, foreign exchange, equation of payments, such as are found in business mathematics, in forming their business judgments. The man of business and finance, town and city officials, corporation heads, and others must be able to read fiancial statements and statistics, and to evaluate the result of policies. Their number is estimated at 300,000, although this knowledge has by no means become widespread.

VIII. Mathematics in Science

There is a group of students and professional men who find mathematics necessary in the study of the various sciences, such as architecture, physics, chemistry, biology, botany, agriculture, zoology, anatomy, bacteriology, optics, and others. These are estimated at 50,000.

IX. Mathematics in Recreation

There are some who find recreation in mathematics—those to whom puzzle columns in newspapers, magazines, almanacs, etc.,

are addressed (Sam Lloyd puzzles). This group is probably rather small, yet there is always the constant interest and demand. Such persons probably rank high in native intelligence. No attempt is made to estimate their number. This group cannot be called essential, and with the objective *recreation* it is doubtful if recreation experts would recommend that public money be spent for increased proficiency in this direction. This should not be confused with teaching method—mathematical recreations will certainly be used in teaching for the stimulation of interest, yet with quite different objectives in view.

X. Teachers of Mathematics

The group of teachers of mathematics. The justification of this group depends on the settlement of curriculum objectives and values to society. It is a group high in intelligence and its present size may be estimated at 25,000.

XI. Mathematics for Discipline

There is at present a large group to whom mathematics is taught as a means of giving command of scientific method or reasoning ability, or inculcating ideals of thrift, accuracy, neatness, thoroughness, etc. It is doubtful whether, if these values were directly sought, mathematics would be chosen as the medium of approach.

XII. Mathematics in Beauty

Mathematics is the foundation of beauty—and its forms are used in architecture, church decoration, and patterns of all kinds. Symmetry is a fundamental element in beauty. The circle, ellipse, square, and star are forms freely used in art. But it is doubtful whether any class or group of individuals use these facts except a very few designers ,architects, etc. They may be numbered at 5,000.

XIII. Mathematics in Religion

For a few the concepts of mathematics, such as the infinite, have a religious significance. This group is undoubtedly small and consists of men who are already grounded in mathematics for other reasons. As such abstract conceptions are merely contemplative and rarely functional, it is questionable whether mathematics need be fostered for this purpose.

Besides these "uses" of mathematics, there is a general diffusion of appreciation among the educated of what mathematics has done in the past and the place it plays in modern civilization, in beauty, in nature, etc.

Curriculum Proposals

The foregoing states the various uses of mathematics in society at present. Probably no effort should be spent in training for the objectives of group IX, XI, XII, and XIII. In these groups mathematics is only a means of achieving some other end, and it would have to be shown that mathematics is the best means to the accomplishment of these ends before they can serve as objectives for mathematical study. That is, it must be shown that mathematics is important to recreation or discipline ,or that it functions in religion or that a knowledge of it is of great use in the production of beauty.

Groups III and IV are specialized and vocational and should be cared for in vocational education. But a foundation of the arithmetical processes must be laid in the grades. Really these groups do not learn mathematics at all, but only the mechanical applications of mathematics—the principles underlying the processes need not be taught.

Groups V and VI are or should be universal and minimum standards and more detailed objectives should be set up.

Groups I, II, VII, VIII, and X have a more or less general use for mathematics, and these groups are the ones to be provided for by our schools and colleges. Nearly all of these go on to higher education, and there is a question as to whether all of their mathematics can be left until the choice of a vocation is made, or whether some of it should be learned before a definite choice is made. Probably there is no psychological reason why such a choice need be made at any specific age. But it is a great advantage to have the tools in readiness when the vocation is decided upon.

However, there are a few ways of sifting out those who are possible candidates for these important and essential posts in society. In the first place, they are of high general intelligence almost without exception. Moreover, they all have powers of analysis or generalization far above the common. The

general intelligence measurement problem is being successfully solved at present, and a little has been done on the prognostic test. It would be possible, then, to single out with considerable accuracy those who would occupy these important posts and strongly to advise them with regard to an early choice. This would by no means exclude those who would also choose to elect this work and who might achieve a measure of success through diligence.

There have emerged three great divisions into which mathematical instruction should fall. (1) Mathematics that one uses as consumer in everyday living and that is necessary to correctly evaluate and judge society. This should be for all. (2) Mathematics used in the various vocations, usually as skills or habits. This is very specialized for each of the many lines of activity. (3) Mathematics for the essentially mathematical worker, with ability to reason in new situations.

Just a word in closing with reference to the mathematics in the junior high school, which is now the storm center of curriculum changes. The junior high school may disregard (2), since that belongs primarily to the vocational school. A course in (1) might be required of all. Such a course in consumer's mathematics would include topics such as personal accounts, meter reading, budgets, making change, discount, bills, interest, banking practice, checks, promissory notes, reading maps and drawing to scale, interpretation of graphs, train fares, parcel post rates, express rates, insurance, building and loan associations, taxes, etc. Special courses might be given to special groups such as girls (household accounts, budgets, heating expense, buying, food values, ventilation etc.), or for children in farm communities (farm accounts, farm records, planting, feeding, soils, roads, etc.). Again a course might be given for all on the forming of social judgments on facts. Some of the elementary statistical concepts would be taught such as the different averages, the frequency distribution, variation, correlation, etc. Instruction would be given against basing judgment on subjective standards. Effort would be made to enable future citizens to evaluate and criticize social policies, acts of legislation, the news, etc. And in all this, the policy should be prodigality rather than parsimony, as social conditions are rapidly changing and it is impos-

sible to foresee possible arithmetical needs of the common man in the future.

Finally a course in (3) should be offered as an elective, open to any who might wish to choose it, and to which those with suitable talents would be strongly recommended. Such a course or courses would be given with the idea of use or functioning all through life. In algebra such a course would include such topics as the formula, the graph, positive and negative numbers and the equation, also intuitional geometry (or demonstrative when facts can be more easily and surely verified by the student), and numerical trigonometry.

Now in such a course the wise teacher will seek to include all the other values of mathematical instruction as by-products. He will endeavor to raise an appreciation of the place of mathematics in the life about us; he will show the recreational side that mathematics offers; he will, where possible, show where mathematics is the foundation of all forms of beauty; he will show how mathematics helps us to comprehend some of the eternal values of life; he wil develop correct habits of thought and work.

A NOTE ON THE FAILURE OF EDUCATED PERSONS TO UNDERSTAND SIMPLE GEOMETRICAL FACTS

By Professor EDWARD L. THORNDIKE
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The following was given, as the third in order of a set of eight paragraphs to be read, to twenty-seven adult students in a Summer Session course in Mental Tests.

Read this paragraph. Then read the questions. Answer them, reading the paragraph again as much as is necessary.

Polygon means in geometry, a figure enclosed by any number of lines—the sides—which intersect in pairs at the corners or vertices. If the sides are coplanar, the polygon is said to be "plane"; if not, then it is a "skew" or "gauche" polygon. If the figure lies entirely to one side of each of the bounding lines the figure is "convex"; if not it is "concave." A "regular" polygon has all its sides and angles equal; if the sides and angles be not equal the polygon is "irregular." Of polygons inscriptible in a circle an equilateral figure is necessarily equiangular, but the converse is only true when the number of sides is odd. The term regular polygon is usually restricted to "convex" polygons.

1. (a) What does "coplanar" mean?................................
   (b) Are all regular polygons convex?..........................
2. (a) Are all polygons either plane or skew?.......................
   (b) Are all polygons either convex or concave?....................
3. (a) Are all equilateral polygons inscribed in a circle regular?........
   (b) Are all regular polygons equiangular?........................
4. Draw an irregular concave polygon that is a quadrilateral with sides approximately 2, 2, 1½ and 1½ inches long.
5. What do we commonly call a regular four-sided polygon?............
6. (a) What figure would be made by placing six regular three-sided polygons of equal area so that six vertices, one from each polygon, met in a common point and so that there was no overlap of area? ....
   (b) What figure would be approximated by a regular polygon of **n** sides as **n** is increased?......................................

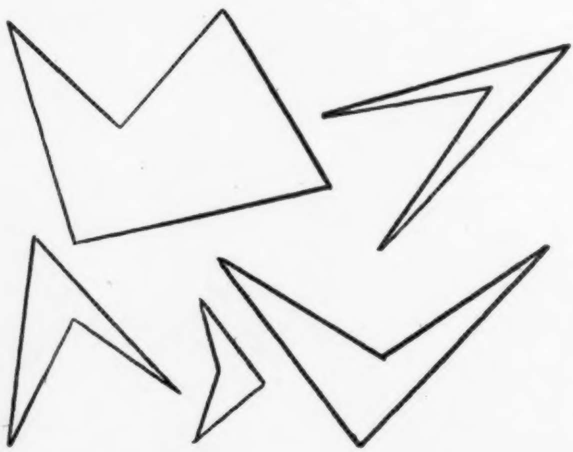
Sixty minutes were allowed in all, and such directions were given as to make it probable that from six to ten minutes were spent on this paragraph. It is highly probable that nine out of ten of the individuals tested had studied geometry for a year or more. It is also highly probable that they were above the average in their high school classes in mathematical ability, general scholarship, and interest in geometry. Such being the case, the frequency, variety, and gravity of their errors in questions 1, 4, 5, and 6 of this task* seem instructive to teachers of

* The questions of 2 were answered correctly by 20 of the 27. The questions of 3 were answered correctly by 21 of the 27. Nothing will be said further about them.

mathematics, though just what they should teach us is not so certain.

Eight, or less than one in three, answered 1*a* correctly. Seven omitted it. The other twelve gave "plane," "flat," "plane polygon," "polygon with plane sides," "sides of a plane polygon," "parallel," "regular," "at right angles," "sides intersect in pairs at corners," "regular, opposite parallel," and "together as a plane." It seems that a clear notion of a plane in condition for ready use as an instrument of thought was absent from the minds of over a third of this group. Any working mastery of "co" was probably as rare, or rarer.

Number 4 was omitted by eleven and drawn as convex* by eleven. Of the five who drew a concave polygon, one used five sides and two made the sides far shorter than the specifications, as shown in figure 1.



It appears that less than one in five of the group had the ability to read and apply quickly the perfectly clear definition of concave polygon given in the paragraph.

Number 5 was answered correctly by thirteen of the group and half correctly by one ("square or rectangle"). The others gave such answers as "quadrilateral," "cube," "trapezoid," "parallelogram," "convex," "rectangle," "oblong," and "op-

* Rectangles, trapezoids, a hexagon and a cube were among these.

posite sides equal." It thus appears that only half of the group have been equipped by their study of geometry with the ability to understand and apply readily, "A regular polygon has all its sides and angles equal"; and this would seem to imply that their actual working knowledge of line, angle, side, and polygon was very weak. At least they did not in the test do much that was useful with them.

Number 6*a* was answered correctly* by fourteen. Two others showed some geometrical appreciation by answering "pyramid." The remaining eleven either omitted 6*a* or gave "rectangle," "circle," "prism," or "equilateral triangle."

Number 6*b* was answered correctly by twelve. Thirteen omitted it. Two wrote "rectangle."

Considering 1*a*, 4, 5, 6*a* and 6*b* as a total, six individuals had none right, seven had one right, five had two right, five had three right, two had four right, and two had five right. Thus it seems conservative to say that two out of three lacked an adequate, fluent, *working* knowledge of what a "point," a "line," an "angle," a "polygon," a "quadrilateral," "approximate," and "*n* sides" mean. They could not use these meanings effectively in slightly new and complex problems. The conclusion seems unavoidable that the common expectations from the study of geometry are pitched far too high. Whether this is due to inadequate teaching or to the essential difficulty of the concepts and terms and relations involved, I shall not try to decide.

* Not fully, however. Only one of the entire group noted that the figure would be a regular hexagon.

TEACHING PUPILS HOW TO STUDY MATHEMATICS

By ALFRED DAVIS
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Continued

An investigation with the high school pupils of St. Louis indicates that there is probably not so much difficulty with the study of algebra as there is with geometry. At least the pupils say they enjoy algebra more. There may not be the need for a detailed discussion of the teaching and study of algebra that there was in the case of geometry. However, algebra should be constantly related to the study of arithmetic. This will make many of its difficulties clear to the pupil who finds trouble in it. We spend too much time on the formal processes as such. Just as it was discovered that the best way to learn the alphabet was incidentally through reading, and not as a thing of importance in itself; so, much of the formal work in algebra is best learned through the thought or translation problems. The writer has heard teachers contend that elementary algebra should be confined almost wholly to the mastery of, and drill in, the formal processes. These processes must be mastered, but much of the motive for work, and the maintenance of interest, can best be achieved through its use in real and applied problems—problems which have a vital interest for the pupil although they may have no great value in adult life. Of course no one will question that the ultimate outcome of an elementary course ought to be a thorough knowledge and mastery of the principles of the subject along with skill in their application; and that these results ought to be achieved in the shortest and most attractive way possible. The subject, however, should not be taught by the novice; who, for fear he could not teach anything else, is given a class in algebra.

Much depends upon the attitude of the student towards his work. He must be willing to pay the price of success; he must make the effort necessary for the mastery of mathematics. Such is the price of success in any line of effort. The mastery of mathematics points out his deficiencies, and points the best road to success. Ability to study mathematics means, as a rule, ability to be a good student; and a good student has, many

times over, the chance to succeed in life that a poor student has. Professor Schultze, in "Teaching of Mathematics in Secondary Schools," says:

"It is a common experience to see a pupil in the upper grades suddenly wake up to the meaning of mathematics and thereby change his attitude towards study in general."

Professor Rietz has shown that the Dartmouth students who excelled in mathematics thereby increased their chances of success in law.

A study by President Lowell, of Harvard University, published in the *Educational Review*, October, 1911, indicates similar results for the Harvard students. Of 609 who graduated from college with A.B. plain, only 6.6 per cent. obtained *cum laudi* in the law school. Students of mathematics attained highest honors in the law school. Students of the classics stood next The qualities of diligence and perseverence, and intensity of application acquired in the study of mathematics secured a higher degree of success than was attained by those who did not study mathematics. If, then, the student finds mathematics hard, let him profit by the determination of Robert Bruce, who, after watching the spider, after many futile efforts finally reach the ceiling, went out and won in battle agains odds. If he is inclined to waste time let him get inspiration from the words of Hotspur before the battle of Shrewsbury:

"Oh, gentlemen, the time of life is short;
To spend that shortness basely were too long,
If life did ride upon a dial's point,
Still ending at the arrival on an hour."

Many of the failures in mathematics are due to indifference and laziness on the part of students. Professor Helen A. Merrill, in MATHEMATICS TEACHER for December, 1918, shows that obstinacy and neglect on the part of students is responsible for some of the failures in mathematics. Students assume the subject hard and refuse to make effort in it. Or it frequently happens that neglect at some point of the course has destroyed the sequence, and the pupil is not willing to make the effort to make up the loss. This is a subject which demands, almost more than any other, constant diligence and effort, alertness on the

part of the pupil that nothing passes by without being thoroughly understood. This is one of the great values of the subject. But the pupil fails to realize the seriousness of the situation; he refuses to make effort. The words of the foreigner who had fallen into the water and called, "I will drown, nobody shall nelp me," apply to him.

An interesting study by President Foster, of Reed College, the results of which he published in *Harper's Monthly* for September, 1916, gives an answer to the question, "Are good students in high school more likely than others to become good students in college?" Three colleges in as many states were considered. Of hundreds of students in the University of Wisconsin, above 80 per cent. of those in the first quarter in the high school remained in the upper half of their classes throughout the four years of their university course, and above 80 per cent. of those who were in the lowest quarter in the high school did not rise above mediocre scholarship in the university. Only one in five hundred of those in the lowest quarter reached highest rank in the university. The University of Chicago found that students who failed to receive in high school an average higher than "passing" by at least one-fourth of the difference between the passing mark and 100 per cent. failed in college; such students are therefore, not admitted; where exceptions are made the record in college is seldom satisfactory. Reed College at its fall opening nine years ago admitted only those students who ranked in the first third in the preparatory schools: about 20 per cent. were exceptions to this rule and 2 per cent. were below the median line; these exceptions were selected as the best below the first third. Of these exceptions, practically none rose above the lowest quarter in their college classes. The same results are shown to be true of those who go from college to the professional schools. Surely "promise in high school becomes performance in college," and the mediocre in high school are practically out of the race. President Foster says:

"If all these studies prove anything, they prove that there is a long chain of causal connections binding together the achievements of a man's life and explaining the success of a given moment. . . . Luck is about as likely to strike a man as lightning and about as likely to do him any good. The best

luck a young man can have is the firm conviction that there is no such thing as luck and that he will gain in life just about what he deserves and no more. . . . Nothing seems to promise failure in the tasks of tomorrow with greater certainty than failure in the studies of today. . . . Among teachers the greatest number of criminals are not those who kill their young charges with overwork, but those who allow them to form the habit of being satisfied with less than the very best there is in them."

Much of the inefficiency of students in mathematics in these days is due to the idea somewhat prevalent in the elementary schools, and even found in the high schools, that a student must gain his education by being amused. He is not trained to feel or to know personal responsibility in the matter. His education is entirely a problem for the teacher, and others, to solve. The pupil even resents any effort to secure his initiative if this would suggest anything like drudgery or hard thinking on his part. If it is true that there can be no real education without interest, it must also be remembered that the pupil has some responsibility in the matter of his own interest. Education involves effort, is in direct proportion to effort, and interest is the product of effort. Any pupil with proper instruction can master the mathematics required of him in our high schools, and mastery will give interest and pleasure.

Regarding the easy drifting which pupils are allowed to do in many of our schools today, Dr. Munsterberg is quoted as saying in the *Metropolitan Magazine*, 1910:

"The community has found out that such schemes may be well fitted to give the children a good time at school but lead them to a bad time afterward. Life is hard work and if they have never learned in school to give their concentrated attention to that which does not appeal to them and which does not interest them immediately, they have missed the most valuable lesson of their school years. It has always been found that it is the general education that pays best and the more the period of cultural work can be expanded the more efficient will be the services of the school for the practical services of the nation."

Professor Bagley, of Teachers' College, says in *Teachers' College Record* for November, 1918:

"There has, indeed, been a tendency in both educational theory and school practice to belittle duty and obligation as motives to effort. . . . There is a tendency, indeed, to go even further than this, and to insist that the learner should not be expected or encouraged—much less required—to put forth effort unless he can see clearly whither the effort leads and why the goal is desirable.

"This extreme view would have a tendency, obviously, to encourage a most unfortunate type of individualism. . . . Under the dominance of this extreme reaction against duty and effort, indeed, good schools have degenerated, and their pupils have suffered a real injustice. . . . To impress upon even young children the notion of education as a democratic duty is not beyond the ability of the skillful teacher. . . . The day of individualism has gone by, and the notion of education as a means of securing an individual advantage in the social order should be and can be replaced by the conception of education as the democratic duty. . . . One need have no fear that the emphasis on education as an obligation of the individual to democracy will encourage the barren formalism which all thoughtful students of education have lamented and striven to remedy."

Another source of trouble is the assumption that when a pupil masters a principle this is a guarantee of his ability to apply it. Such is not a necessary result. Many of our best students appear sadly inefficient when they leave high school because they have not been taught the use of their acquisitions. Application is quite as important a part of our work as the teaching of principles. Of course no pupil can be taught all possible applications of his knowledge, but he can be taught *how* to apply it. Professor C. H. Judd, in "The Psychology of High School Subjects," says:

"It is contrary to experience to assume that students can apply mathematics to the other sciences or to the practical affairs of life unless they are trained to see mathematical relations in other forms than those in which they are commonly presented in the schools. The student who knows the abstract demonstra-

tions of geometry, but does not realize that knowledge of space is involved in every manufacturing operation, in every adjustment of agriculture and practical mechanics, is only half trained. Application must be a phase, and an explicit phase, of school work. Application is as different from pure science as pure sciences are different from each other."

The study of mathematics in our high schools could be greatly facilitated by a more skillful classification of pupils. All normal pupils have ability to study mathematics if the subject is properly taught. Those who cannot study the subject are probably no more numerous than physical defectives, and in many cases may belong in special institutions. Every one needs to study mathematics to be ordinarily intelligent on many topics of common interest; yet there are, without doubt, pupils who are not as apt as others in the study of the subject. Pupils can probably be divided into three fairly distinct classes, for purposes of instruction; those of more than ordinary ability; those of medium ability—the largest class; and those below normal. At the present time we are obliged to take pupils of all grades of intelligence and teach them the same things, in the same way, in the hopes of getting the same results from all. This is, however, impossible. Mr. Gruenberg, in *School and Society* for February 15, 1919, says:

"It is desirable that teachers generally learn to think in terms of individual variation, and in terms of distinguishable capacities—a very different attitude from the traditional one heretofore cultivated."

The classification of pupils into three groups might be the work of an expert psychologist. The work of the psychologists in classifying men for service in the recent war might be extended to the schools with advantage. Provision should be made for re-examination and re-classification into higher groups as the pupil develops. The differences within such groups should not be so great as to discourage anyone; indeed there should be an incentive for all in the possibility of promotion to a more rapidly moving group. As conditions are, if we set the pace for the medium group both the other groups suffer, if we set it for either of the others matters are much worse. Mathematics probably suffers more than any other study from this situation.

Some improvements are being made, chiefly in the large centers and in junior high schools, but scarcely a beginning has been made.

In some localities the importance of teaching pupils how to study is recognized by the introduction of supervised study. One of the chief difficulties with supervised study lies in the fact that teachers themselves do not know and are not instructed how to use to advantage the extra time given a class for this purpose. Some teachers consider it merely an opportunity for the pupil to study under favorable conditions, where home conditions are not well suited for study. Others think of it as a time for the pupil to consult the teacher for help when he finds a problem which he cannot do, and the pupil is assisted with this difficulty. Yet others look upon it as a time when the school authorities wish the pupils to work, while the teacher has opportunity to do work of his own, read a story, etc. And others use it as extra time for teaching the subject. If these are the only uses made of the extra time, then supervised study is a complete failure. The mortality of the classes may be reduced, but this may mean nothing. The teacher merely feels that the pupil has done the best that could be expected of him, that he has spent sufficient time studying, that he would probably gain little from repeating the course, and that the supervised study period is intended to reduce failures, hence the pupil is passed on without due regard to scholarship. Such procedure involves the pupil in endless difficulty later unless the same policy is used throughout his student career. However, to tell a pupil he is educated when he is not educated is to give him false values of life, it is to encourage Bolshevism. The schools with such a policy would become a menace to, rather than a guardian of, our national interests.

In the proper use of a supervised study period the teacher must first of all know how to study, he must then teach the class how to study. The pupils should then apply the information and training given them to the lesson in hand. Some weak members of the class may need individual assistance, but this should be given in such manner as to teach the pupil how to do his work without help. The pupil must not feel dependent on the teacher while he studies. He should study and master the

lesson as he will later meet problems out of school which must be mastered. In the case of pupils who may have learned how to study, the teacher must not interfere with their preparation of the next lesson. Such pupils may be used to the advantage of the other members of the class. They may suggest effective methods for study. However, if the teacher knows how to teach, he will be constantly teaching how to study as he teaches the subject. In such case there will be no need for a supervised study period, except for backward pupils, and for those who have home conditions unsuited for study. After a lesson has been properly taught, and if the pupil has given his attention and interest to the work, he should feel that further work assigned on that topic is purely his own personal responsibility. Unless, then, supervised study periods are properly used we can get along better without them.

We are indebted to Mr. S. A. Douglass, Principal, Central High School, and Principal of the Summer Session of the Soldan High School, St. Louis, for the following information concerning an experiment which he made in the summer of 1916. We give a brief account of it here because it emphasizes the importance of teaching pupils how to study, and it contains valuable suggestions for those interested in helping to solve the problem.

A printed card was given each pupil. The front of the card contained blanks for marks on recitation, study, and standing, to be filled in by the pupil each week, and these were to be checked by the teacher, who made such suggestions as he thought necessary. The pupil was also to indicate place of study, time of study, and amount of study. The back of the card contained the following excellent suggestions on how to study:

I. Conditions Favorable to Successful Study.

1. Study away from interruptions as far as possible.

2. Have a definite study program and follow it faithfully. Give to each subject its just share of your study time.

3. Study your lesson as soon as possible after the assignment is made.

4. Concentrate your mind so that outside interests will not frequently disturb your study.

5. The use of the dictionary and reference books promotes good work.

II. Attacking the Lesson

1. Make sure that you clearly understand the subject of the lesson, or the particular problem to be solved.

2. Find the important facts in the new lesson and connect them with the facts previously learned.

3. Group the minor points of the lesson about the leading topics, thus making an outline of the work in hand.

4. Do not try to commit exact words until you understand the content of the sentence or paragraph. Mechanical memorizing is never advisable.

5. Make comparisons and contrasts when possible.

6. Review frequently. This greatly aids in the assimilation of knowledge. Reflective thinking is eminently worth while.

7. Make up your mind that you can learn. A strong will can accomplish wonders. Difficulties fade away for the person with an unflinching determination.

III. Meaning of the Marks to be Used

"E" means that you have grasped the subject; thought about it; made it your own; so that you can give it out again, with the stamp of your individual insight upon it.

"G" means that you have taken it in and can give it out again in the same form in which it came to you, but the words come from the book or the teacher and not from you.

"M" is much like "G," only that your second-hand information is partial and fragmentary, rather than complete.

At the close of the session the pupils were questioned with the following results:

Out of 1887 responses:

1. 1,536 kept the record; 281 kept it partially; 56 neglected it entirely.

2. 1,456 kept it day by day throughout the term.

3. 1,046 thought the keeping of the record had helped them in some way; 444 were uncertain as to the results; 383 said it was no help.

4. 900 said they had carefully studied and followed the suggestions on the back of the card; 805 had done so partially; 142 had given the suggestions no attention.

5. 875 recommended some such plan for the regular school term; 685 opposed its use in this way; 271 were non-committal.

A few of the answers as to how they had been helped are:

I am able to see whether the week has been successful or not.

The suggestions helped me to grasp things more easily.

It showed me what I was doing and made me work harder.

It gave me a better idea of the teacher's view of my work.

In order to put a good grade on my card I worked harder.

I profited by the suggestions made by the teacher.

It told me whether I was overestimating my results.

It helped me to know how I stood in class.

It gave a more comprehensive idea of work value.

I could tell whether or not I had studied enough.

It helped me to find out what to study most.

It gave me confidence in myself.

It reminded me of my past record, spurred me to study, and made me more careful in my study.

I noticed the ratio between study and recitation marks.

It gave me greater accuracy in the judgment of my work, and in valuing my work.

All the suggestions on the back of the card were recognized as helpful, but the following were emphasized most:

Study away from interruptions as far as possible.

Study your lessons as soon as possible after the assignment is made.

Review frequently. This greatly aids in the assimilation of knowledge. Reflective thinking is eminently worth while.

The teachers were also questioned as to the result of the experiment, with results as follows:

1. Practically all the pupils were perfectly willing to keep the record.

2. Most of the pupils showed care and fidelity in marking the card, some were indifferent and a few were doubtless dishonest.

3. There was only fair evidence that the pupils tried to carry out the suggestions on "How to Study."

4. There was considerable evidence that the pupils do not understand the meaning of marks in the best sense of the term.

5. In general the spirit manifested by the pupils in carrying out the details of the plan was good. Some were indifferent.

6. About half the pupils showed a commendable degree of

accuracy in estimating the worth of their work. The poor and indifferent pupils frequently overestimated the value of a recitation.

7. The very best pupils showed dissatisfaction with the plan, especially the marking part of it. The younger pupils were enthusiastic about the plan, while the older ones were not.

8. The teachers were unable to report any noticeable improvement in the pupils' method of study, although many of the pupils themselves reported that their method of study had improved.

9. The teachers generally recommended the follow-up plan (checking by the teacher) and some thought it absolutely necessary. Personal interviews generally brought results, but in some cases pupils continued to mark themselves too high.

10. The chief objectioon to the cards was that they unduly emphasized marks.

11. The card was thought of value to certain classes of pupils and of little value to others.

12. The best results seem to have been reaped by medium pupils.

13. The best and poorest pupils got little or no benefit; the former because they already had good methods, and needed no stimulus; the latter because of indifference and inate slowness.

14. The teachers thought the suggestions on the back of the card most valuable. The follow-up plan was ranked next in importance.

15. A small majority of the teachers thought it worth while.

16. It was not thought advisable to recommend the plan for the regular term without further trial and study.

17. It was recommended that pupils report the time spent in the preparation of each lesson rather than on the quality of their study.

18. It resulted in a better understanding between teacher and pupil, and a more systematic preparation of the daily work.

19. The suggestions on how to study should be enlarged and made more specific.

The purpose of the experiment seems to have been to determine what might be done towards improving the study habits of pupils. No very definite conclusions seem to have been

reached. The weight of responsibility seems to have fallen upon the pupil rather than upon the teacher. In case the pupil has bad habits of study, or no habits of importance connected with study, it is not probable that the pupil will have either understanding or initiative sufficient to enable him to form good habits. The teacher must be held responsible for teaching him how to study, if any real improvement is to be expected. Such an experiment is of value in emphasizing the importance of the matter, and in suggesting how to attack the problem to advantage. A summer session is too short and too hurried a time in which to apply the remedy effectively. We might almost as well expect the pupil to teach himself as to expect him to place correct values on his own efforts and achievements. It is worth while to know what he thinks, but only for the purpose of assisting him in the most advantageous way.

The fact that the best pupils saw no value in the cards is not necessarily an argument that they did not need instruction in methods of study. They may still fall far short of their possibilities. Many of those who have attained some measure of success in life realize that valuable time and effort might have been saved if they had been taught in the beginning how to study, and that some of the bad habits formed were later almost impossible to break.

Such experiments as this by Mr. Douglass, and all efforts to teach pupils how to study, will eventually result in the successful solution of the greatest problem that confronts us in our teaching at the present time.

Correct methods of study involve the proper use of text books. Every pupil should own the book he uses for a text. Not only this, but the book ought to be kept for future reference. Books are for the purpose of aiding the student to master the subject, and should be used in any manner that will help. They are not to be preserved to bring the highest possible price as soon as the course is ended. They should be underscored, have notes written on the margin, or interlined. In this way they can be made convenient for quick and ready reference, or in any case, they can be made to assist the owner.

It is for this reason that free texts are undesirable. Of course a borrowed book is better than no book but if pupils are unable

to buy texts, a way should be found by the community, by which they can own the books they use. Again free texts are undesirable for sanitary reasons. They are frequently passed from one pupil to another without any attempt to clean them, and in this condition they may be the carriers of disease. Yet again, some of the texts become decidedly shabby before they are discarded, and since a pupil has no choice but to take the shabby book if such is offered him, a strong dislike for the subject may originate in the objectionable book. While there are some strong arguments in favor of free text books, the writer believes the objections to them are much stronger; one of these is, the pupil cannot be taught some of the important points in the art of study if he does not own the book he uses.

The pupil should form the habit of noting the author, title, publisher, and date of publication of the books he reads. He should, if possible, learn something about the author. In this way he can learn to compare authorities and form independent judgments of his own regarding them. He should study the contents to obtain a general notion of the material covered in the book; and he must learn to use the index for purposes of reference. It is amazing how few pupils seem to know that there are such parts to the book, much less that these parts can be of any importance to them. If they have noticed these parts at all it has been with the idea that in some way they were necessary for every book, but that no one thinks of using them.

Text books ought to be up to date; not only that, but they ought to be selected with care as the best available of the modern books. No text in plane geometry is satisfactory which does not provide constructions and informal proofs at the beginning of the course. It should postulate theorems that appear to the average student as self-evident. It should provide numerous applied problems, connecting the subject with other fields; and some reference should be made to the trigonometric functions, etc. No text in algebra is satisfactory which does not relate the work to arithmetic; which does not use the graph; which does not emphasize the function, and the equation; and which does not give wide opportunity for the development of the reasoning powers in thought problems, as well as exercises for the mastery

of technique. We need a better type of book than we possess for mathematics. So far they are dominated by the interests of the subject. We need texts constructed according to the interests of the pupil, since he is the most important factor in the consideration. We read books arranged and graded according to the ability of the pupil to learn the subject. Professor F. M. McMurray, in "How to Study," says that text books should contain abundant detail:

"Without plenty of detail the facts have to be run together, or listed, merely as so many things that are true; they then form only a skeleton, with all the repulsiveness of a skeleton."

It is claimed by some persons that a teacher should be able to teach without a text, or to make his own text. No doubt the teacher should be a thorough master in his subject, but he may not be able to write a text book. The making of text books requires special abilities and few teachers possess them. It is no reflection on the ability of a teacher if he cannot write a text book. Moreover texts should be the outcome of wide experience in the teaching of the subject, it is probable that they can be written better by a group of teachers than by a single individual. It is a waste of time and energy to teach mathematics without a text, and few teachers try it. A poor text is better than none. Professor L. V. Koos has shown as the result of an investigation including 416 schools in 15 states of the North Central Association of Colleges and Secondary Schools, that:

"Text books dominate content and organization of courses in mathematics (in high schools)."

Commenting on the same topic, Rugg and Clark, in "*Reconstruction of Ninth-Grade Mathematics,*" say:

"It seems evident that *text books almost always completely determine the specific subject-matter that is taught to students in the course.*"

The same writers divide high school teachers of mathematics into three groups:

1. The typical mathematics teacher, with no specific training in mathematics beyond a one or two years' course in college mathematics, and no training in teaching mathematics, and only one, two, or three years' experience.

2. A group of teachers in larger cities, who have longer experience, fairly adequate college training in mathematics, but almost no training in the teaching of mathematics. These change the order of subjects and supplement the material of the text.

3. A small group, well trained in content and in professional courses. Many of these are either independently or in connection with college men, textbook writers.

We have seen that the advantages from the study of mathematics cannot be properly realized unless pupils are taught how to study, that this matter is very greatly neglected at present, and that many elements enter into the successful teaching of the art of study as it relates to mathematics. Some attention is being paid to the matter but it still remains our most serious problem.

NEWS AND NOTES

Dean J. H. Minnick, President of the National Council of Teachers of Mathematics, is preparing the program for the third annual meeting of the Council, to be held in Chicago, March 1, 1922, during the week of the N. E. A. It is hoped that the program of this meeting may even exceed the splendid programs of the preceding meetings. The Chicago Mathematic Teachers' Clubs are assisting in arranging for the meeting.

Mr. Alfred Davis, of St Louis, was elected President of the Central Association of Science and Mathematics Teachers for the years 1921-1922. Mr. Walter Gingery, of Shortridge High School, Indianapolis, is Chairman of the Mathematics section.

The New York City sections of the Associations of Teachers of Mathematics in the Middle States and Maryland met December 2, 1921, at the Aldine Club, for a dinner-program. The general topic of the program was "The Next Step in Junior High School Mathematics." The speakers and their subjects were: Professor David Snedden, Prescribed versus Elective Courses in Junior High School Mathematics; Dr. Joseph K. Van Denberg, The Next Step for the Administrator; Professor William H. Kilpatrick, The Next Step in Method; Professor C. J. Keyser, The Human Worth of Rigorous Thinking; Professor David Eugene Smith, The Next Step in Content.

Synopses of these papers will appear in the January number of the TEACHER.

The officers of the Section are: Raleigh Schorling, Chairman and Louise Webster, Secretary.

The Association of Teachers of Mathematics in Southern Massachusetts, met in Fairhaven, Mass., on Saturday, November 5, 1921. The following program was rendered: 1. How Often Should Tests Be Given?; Discussion led by Miss Louise Bullard, Taunton High School. 2. Should there be Tests where Absolute Perfection is Required on a Question? Discussion led by Miss Mary F. Hitch, New Bedford High School. 3. Should Tests Be Too Long for the Pupil of Average Ability to Finish? Discussion led by E. Estelle Miles, Durfee High School, Fall River. 4 Discussion of the Report of the National Mathematics Committee

on "Terms and Symbols in Elementary Mathematics," by Mr. Harry D. Gaylord, Brown and Nichols School, Cambridge, Mass.

The officers of the Association are: President, Mr. Edmund D. Searls, New Bedford; Vice-President, Miss Margaret English, Bourne; Secretary-Treasurer, Miss E. Estelle Miles, Fall River.

The Association of Teachers of Mathematics in New England held its nineteenth annual meeting at the Massachusetts Institute of Technology, on Saturday, December 3, 1921. The program follows: Geometry Understood, not Memorized, by Mr. Roland R. Smith, Newton Classical High School; Modern Methods in Junior High Schools, by Mr. Eugene R. Smith, Head Master, Beaver Country-Day School, Brookline, Mass., and Head Master, Park School, Baltimore, Md.; Graphic Solutions of Certain Systems of Equations, by Professor Robert E. Bruce, Boston University; A Geometric Picture Film, by Mr. Charles H. Sampson, Huntington School, Boston.

(This film is not an attempt to teach Geometry by means of moving pictures, but rather the beginning of an effort to create an interest in the subject by using moving picture films.)

The officers for 1921 are: Walter F. Downey, President, English High School, Boston; Professor Clinton H. Currier, Vice-President, Brown University; Harry D. Gaylord, Secretary, Brown and Nichols School, Cambridge. (Address, 448 Audubon Road, Boston), and Harold B. Garland, Treasurer.

The Mathematics Section of the Wisconsin State Teachers' Association met in Milwaukee on November 3. The program consisted of: Improvement of Instruction in Algebra, W. J. Osborn, Department of Public Instruction, Madison; The Mathematics of Life Insurance, L. W. Dowling, University of Wisconsin; The Work and Aims of the National Council of Teachers of Mathematics, Mary A. Potter, High School, Racine; Concrete Geometry in the Junior High School, W. H. Fletcher, State Normal School, Oskosh; discussion led by Mrs. Rose Bruins, High School, Racine.

The thirty-seventh meeting of the Teachers of Mathematics in the Middle States and Maryland was held Saturday, November 26, at Swathmore College in affiliation with the Association of Colleges and Preparatory Schools.

At the morning session a discussion of the new College Entrance Requirements in Mathematics was led by Dr. Jonathan T. Rorer of the William Penn High School, Philadelphia, Dr. H. Ross Smith of the South Philadelphia High School for Boys, Philadelphia, and Miss Elizabeth F. Johnson, of the Baldwin School, Bryn Mawr.

Dr. Rorer emphasized the desirability of having the college units reflect accurately the time needed. He also urged the inclusion of constructive geometry. He heartily supported the tentative report of the National Committee and felt that on the whole the college requirements were being made harder except from the point of view of the exceptionally trained and enthusiastic teacher. Mr. Smith felt pleased in seeing little or no emphasis on the function concept. He deplored the omission of specific emphasis on literal equations. Miss Johnson made a strong plea for pure mathematics. Applied mathematics, she contended, cannot exist unless pure mathematics is kept well in advance of it. The discussion was then taken up by Professor Chambers, Mr. Betz, Dean Hawkes and Professor Young.

Mr. Ernest H. Koch of the High School of Commerce, New York City, gave a report on the survey made by the Mathematics Department of the New York Society for Experimental Study of Education cooperating with the National Committee of Mathematics Requirements. The report was a composite of the answers to a questionnaire sent to representative college professors and another sent to representative leaders of industry. They were asked to indicate the use for and the need of various topics in mathematics and the relative importance that they would attach to such topics. Although the results indicated a slight shift of emphasis, on the whole the report supported the proposed secondary school course in mathematics.

In the afternoon session, Mr. W. E. Breckenridge of Teachers College, New York City, spoke on the slide rule in mathematics courses. He dwelt on three points: (1) Recent developments. (2) A few points on theory and (3) Equipment.

Professor J. W. A. Young of Dartmouth College, Chairman of the National Committee on Mathematics Requirements, gave an inspiring paper on Ninth Year Mathematics. He spoke of

the aims and of the fact that ninth year work could and should be conducted independent of college requirements. He felt that the ninth year should include a more comprehensive program than the traditional course in algebra alone.

The meeting was concluded by transaction of business which included the adoption of an amendment to article 9 of the constitution and the adoption of the amendment that the General Association should ask for twenty-five cents per member from each section for the expenses of the parent association. There followed the election of officers for next year. Director C. B. Walsh of the Friends Central School, Philadelphia, was elected President of the Association.

The fifteenth annual high school conference of the University of Illinois was held at Urbana, November 17-19, 1921. About 3,000 persons attended.

Among the sixteen sections of the conference, the Mathematics was one of the largest. Over two hundred registered in this section. The program was chiefly the work of C. M. Austin, of Oak Park, Ill. There were two sessions, each three hours in length. At the morning session, Mr. R. L. Modesitt, of Charleston, gave a scholarly paper on Fundamental Principles of Algebra. Miss Emma C. Ackerman, of Lockport, aroused much interest in the Proper Amount of Drill in Elementary Algebra. Mr. E. W. Schreiber, Maywood, and Mr. C. M. Austin, Oak Park, reported a study in their schools to determine whether there is any relation between general intelligence and ability to high school algebra. They found a high correlation between achievement in algebra and I. Q. Mr. H. C. Wright, Chicago, gave some of the details of teaching algebra to freshmen without assigning home work. In the afternoon, Mr. G. A. Harper, Kenilworth, read a paper on Experimental Geometry—What? Where Taught? Miss Lida C. Martin, Decatur, reported an experiment in testing mathematics teaching in several Illinois high schools. The testing had led to improved results in several schools. The program was concluded by a symposium on the report of the National Committee on the Reorganization of the First Courses in Secondary School Mathematics, by J. T. Johnson, Chicago; Jessie D. Brackensiek, Quincy; Ruth Utley, Rockford; J. K. Mc-

Donald, Decatur; Grace E. Madden, Champaign, and George H. Fischer, St. Charles. The speakers expressed approval of the general mathematics proposed by the National Committee.—(H. C. Wright.)

The Use of the Slide Rule Permitted on Regents' Examinations in New York State. The State Department of Education of New York State requests that publicity be given to the following statement: "In the examinations in Trigonometry and Intermediate Algebra, the use of the Slide Rule for checking is permitted, provided that all computations are shown on the answer paper." It is the intention of the State Department that pupils using the Slide Rule proceed as follows:

1. Work the problem by the use of the tables.

2. Check the problem by working it on the Slide Rule.

3. If the Slide Rule shows an error, search for the error and correct it.

4. If the Slide Rule shows no error, and if time allows, make an additional check by logarithms or natural functions.

If the question paper calls for a check, it will not be sufficient to say "Checked by the Slide Rule." The pupils will be required to show on the answer paper a check by logarithms or natural functions. However, the use of the Slide Rule will save time in detecting and locating any error that may exist.

The Mathematics Teacher

DEVOTED TO THE INTERESTS OF MATHEMATICS
IN JUNIOR AND SENIOR HIGH SCHOOLS

VOL. XIV DECEMBER 1921 NUMBER [illegible]

Published by the

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

AT 51-53 MAIN STREET, YONKERS, N. Y.

NEW YORK [illegible]

Entered as second-class matter November [illegible], 19[illegible], at the Post Office at Yonkers, N. Y., under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in section 1103, Act of October 3, 1917, authorized November [illegible], 1921.

MATHEMATICS TEACHER

THE OFFICIAL JOURNAL OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

This organization has for its object the advancement of mathematics teaching in junior and senior high schools.

All persons interested in mathematics and mathematics teaching are eligible to membership. All members receive the official journal of the National Council—MATHEMATICS TEACHER,—which appears monthly, except in June, July, August and September.

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Correspondence relating to subscriptions, advertisement, and other business matters should be addressed to

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